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A time dependent thermal model for a superconducting constriction based weak-link (WL) is discussed for investigating the deterministic dynamics of its temperature and phase. A new dynamic regime is found where a non-zero voltage exists across the WL, and its temperature stabilizes between the bath temperature and superconductor's critical temperature. This regime exists over a limited bias current range and gives rise to a new hysteretic regime in current-voltage characteristics. We also discuss the effect of fluctuations on the current-voltage characteristics and experimental implications of this dynamic regime. © 2014 AIP Publishing LLC.
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INTRODUCTION

Superconducting weak-links (WLs) acting like Josephson junctions (JJs) [1] have been used in micron-size superconducting quantum interference devices (μ-SQUIDs), which in turn have been used in probing magnetism at small scales.[2,3] Thus, the physics of such WLs has been of interest and efforts have been made to understand their current-voltage characteristics (IVCs). The JJ like current-phase relation of the WL, which dictates the critical current variation with external flux in SQUIDs, is well understood using Ginzburg-Landau approach.[1] However, the understanding of other detailed features in IVCs, particularly, hysteresis, its temperature evolution, and ways to eliminate it, in order to improve μ-SQUID's performance, needs further work.

Hysteresis also exists in the I-V characteristics of the conventional JJs due to large junction capacitance (C). This is modeled by resistively and capacitively shunted junction (RCSJ) model,[4,5] and thus, the hysteresis in these JJs can be eliminated by using a small-value shunt resistor (R) in parallel with the JJ. In the non-hysteretic regime of these JJs, for bias-currents higher than I_c, the current is dynamically shared between the shunt resistor and the JJ. In WLs, the hysteresis is observed at low temperatures, while at higher temperatures, it is found to disappear.[6–9] Two approaches regarding hysteresis in these WLs have been proposed: (1) hot-spot model[10] and (2) RCSJ like approach but with an effective time constant, similar to capacitor charging time RC of RCSJ model, but related to the recovery of the superconducting order parameter in the WL.[11] Direct evidence of hot-spot in some WLs has been found in hysteretic regime.[12] The second approach explains the I-V curves in the hysteretic regime by asserting that the superconductivity recovery (or Cooper-pair relaxation) time is larger than typical phase evolution time. Kramer and Watts-Tobin,[13] using time dependent Ginzburg-Landau (TDGL) equations, found hysteretic IVCs in superconducting filaments due to oscillatory phase-slip solutions. The hysteresis dominates in certain parameter regime[14] where relaxation time of the magnitude of the order-parameter, i.e., $\tau_{|\psi|}$, is greater than that of the gradient of its phase, i.e., τ_ϕ. The non-hysteretic I-V curves,

observed at higher temperatures, using this approach have often been modeled using an over-damped RCSJ model.[7]

The hot-spot model was proposed as a static thermal model,[10] where temperature profile near the WL is assumed to be time-independent for a given bias current. While the features in hysteretic regime are satisfactorily understood with the static thermal model, it seems reasonable that a non-hysteretic IVC, seen at higher temperatures,[6–9] will have a current switching between the resistive and non-resistive branches in a dynamic fashion as is the case with the RCSJ model or with phase-slip processes.[13] A phase-slip can amount to cooling[15] or heating[16] of quasi-particles in the WL, affecting its current-voltage characteristics, but it will always lead to a net heating in the region near WL over the branch-imbalance length.[17] The thermal dynamics due to phase-slips and its effect on the order-parameter evolution have not been explicitly investigated by such time-dependent models. In one model, by Vodolazov et al.,[18] heat evacuation was analyzed, but the effect of thermal relaxation time-scale on the evolution of the order parameter magnitude was not described. Shah et al.[19] proposed a time-dependent thermal model for the stochastic phase-slips and evacuation of heat thus generated in long superconducting nano-wires. The dynamics of local temperature was thus analyzed, and it was found that in one regime, phase-slips can cause a thermal runaway while in another regime, one can have occasional phase-slips with some local heating but without a runaway. Thermal stability of local superconducting order in presence of super-current is of great practical importance, in particular for SC-magnet wires and cables, helium level sensors, radiation-detectors,[20] SC-WLs and other small scale SC structures.[21,22]

In this paper, we discuss a similar time-dependent thermal model to investigate the deterministic dynamics of phase and temperature in a superconducting WL. We find a new dynamic regime, as a result of the heat balance between heat evacuation and generation. In this regime, superconducting phase-difference across the WL evolves continuously giving rise to a DC voltage and raising the WL temperature but without a thermal runaway. A fraction of bias-current, on

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average, is still carried as the super-current in this regime. This regime exists over a limited bias-current range and gives rise to a new thermal hysteretic regime. Effect of fluctuations and experimental consequences are also discussed briefly.

TIME DEPENDENT THERMAL MODEL

We first recall the static thermal model as proposed by Scockpol et al.[10] In this model, when the current is ramped down, across a superconducting constriction (see Fig. 1), from a value above its critical current (I_c), a static hot-spot, with temperature greater than T_c, is sustained near the constriction down to a current, which we call as static re-trapping current, I_{sr}. This defines a bistable region (for $I_c > I_{sr}$) in the I-V characteristics in bias current range $I_{sr} < I < I_c$ with one branch having zero voltage and the other a finite voltage. The finite voltage branch is found by solving the static heat conduction equation so as the heat generated by the resistive hot spot, at temperature greater than T_c, is conducted away to the substrate or the bulk superconductor nearby. For constriction-dimensions less than thermal healing length, $\eta = \sqrt{\kappa t/\alpha}$, whole of the constriction stays above T_c, in resistive-state, and the $T = T_c$ interface occurs in the bulk electrodes. Here, κ is thermal conductivity of the film, t is film thickness, and α is the heat-transfer coefficient between film and the substrate kept at bath temperature T_b. As the bath temperature rises, both I_c and I_{sr} decrease, but with a different temperature dependence, and thus, the two cross each other at some value of the bath temperature, $T = T_h$ (Ref. 6) giving rise to a non-hysteretic behavior above T_h.

We consider the same superconducting-constriction as a WL, between two bulk superconductors (see Fig.1), with its dimensions smaller than η (Ref. 10) so that the whole WL can be assumed to be at the same temperature T. When a bias current (I) flows through the WL, its instantaneous temperature (T) can be larger than the bath temperature (T_b) giving rise to the heat flow from the WL to the bath via the interface with the substrate or via the bulk superconducting electrodes on the two sides. The temperature will relax over the length scale η in the two superconducting electrodes.

Another consequence of this heating is the reduction in the critical current, $I_c(T)$, of the WL. The bias current, I, will be dynamically shared between normal resistance, R_N, and the super-current (I_s), thus $I = (V/R_N) + I_s$. We also assume R_N to be independent of temperature. The resistivity at such
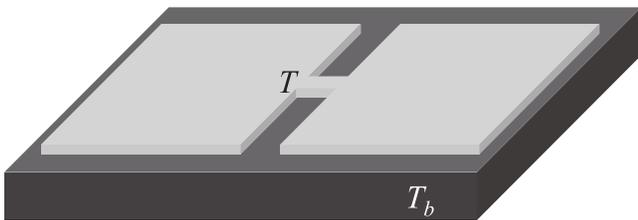

FIG. 1. Schematic of a superconducting constriction based WL on a substrate held at bath temperature T_b. The constriction's instantaneous temperature (T) can be different from T_b depending on the bias current I_b as discussed in the text.

low temperatures is independent of temperature, so the main assumption here is the confinement of the hot-spot to the constriction. The dynamic current sharing, here, is quite similar to the RCSJ model with the additional fact that the heat generated in resistive shunt heats up the WL. A time dependent voltage, $V(t)$, arises across the WL whenever ϕ, i.e., the phase difference between the two superconductors across the WL, changes with time. This is described by the ac-Josephson relation, $V = (\hbar/2e)d\phi/dt = (\Phi_0/2\pi)d\phi/dt$ with $\Phi_0 = h/2e$ as the flux quantum. This voltage V drives a normal current through the WL. We assume a (quasi-static) diffusive limit so that the mean free time for electron scattering is smaller than the time scale over which V or ϕ changes. We further take a short WL limit so that super-current (I_s) and ϕ relation is sinusoidal, i.e., $I_s = I_c(T)\sin\phi$.[1] Thus, at a given instant of time, the total bias current shared between resistive current and super-current branch is given by

$$I = \frac{V}{R_N} + I_c(T)\sin\phi. \tag{1}$$

We further assume that the rate of heat evacuation from the WL is linear with ($T - T_b$). With k as an effective heat loss coefficient independent of temperature, we write the heat evacuation rate as $k(T - T_b)$. The heat loss from the WL occurs through the bulk electrodes and also via the interfaces. As we saw earlier, a time evolving ϕ gives rise to V, which enforces a resistive current giving rise to heat generation in the WL at a rate $V^2/R_N = (\Phi_0^2/4\pi^2 R_N)(d\phi/dt)^2$. The rate at which the WL temperature changes with time is governed by the net rate of heat accumulation in the WL and the heat capacity of the WL, i.e., C_{WL}. We assume C_{WL} as temperature independent. Thus, the expression governing the evolution of the WL temperature is given by

$$C_{WL}\frac{dT}{dt} = -k(T - T_b) + \frac{V^2}{R_N}$$
$$= -k(T - T_b) + \frac{\Phi_0^2}{4\pi^2 R_N}\left(\frac{d\phi}{dt}\right)^2. \tag{2}$$

We note that the branch-imbalance and related length scale will also affect C_{WL}. When the current is carried by quasi-particles across the WL, their relaxation to Cooper-pairs and subsequent heat generation will occur[17] over branch-imbalance length going beyond the WL. This will effectively increase C_{WL}.

The heat generation can also be looked at as the release of energy with each phase-slip event, i.e., a phase change of 2π. These phase-slips may occur at certain rate. The energy released in each phase-slip event, at a given bias-current, I, is given by $(\Phi_0/2\pi)\int_0^{2\pi} I d\phi$, i.e., $I\Phi_0$. The time-scale, over which this phase-slip occurs, is of order $\tau_J = \frac{\Phi_0}{R_N I_c(T_b)}$, which we shall call as Josephson time. Thus, the energy released in each phase-slip is independent of R_N but the rate of phase-slip depends on R_N. Here, we have also assumed that the charge-imbalance time (associated with quasi-particle to condensate conversion or vice-versa) and the Ginzburg-Landau time (which describes the relaxation of superconducting order-parameter) are smaller than τ_J.

We further simplify above equations by defining a dimensionless normalized temperature $p = (T - T_b)/(T_c - T_b)$,

thermal time constant $\tau_{th} = C_{WL}/k$ and normalized bias current $i_b = I/I_c(T_b)$. We also non-dimensionalize time using τ_J by defining $\tau = t/\tau_J$. Thus, from Eqs. (1) and (2), we get

$$\dot{\phi} = 2\pi[i_b - i_c(p)\sin\phi] \qquad (3)$$

and

$$\alpha\dot{p} = -p + \frac{\beta}{4\pi^2}\dot{\phi}^2. \qquad (4)$$

Here, $i_c(p) = I_c(T)/I_c(T_b)$, $\alpha = \tau_{th}/\tau_J$, and $\beta = \frac{I_c^2(T_b)R_N}{k(T_c - T_b)}$. For most experimentally studied devices, it turns out that $\alpha \gg 1$, i.e., $\tau_{th} \gg \tau_J$. The bath temperature dependent parameter β is an important parameter as it controls the phase and temperature dynamics. We can interpret β by writing it as a ratio of two time scales in two different ways: (1) $\beta = \tau_{th}/\left(\frac{C_{WL}(T_c - T_b)}{I_c^2(T_b)R_N}\right)$ or (2) $\beta = \left(\frac{2\pi E_J}{k(T_c - T_b)}\right)/\tau_J$ with $2\pi E_J = \Phi_0 I_c(T_b)$. In the first case, it represents the ratio of the heat conduction time (τ_{th}) to the time required for heating the WL by $\Delta T = T_c - T_b$ using the resistive Joule power at critical current. The second expression represents the ratio of the time taken to conduct away the heat generated (at $I = I_c$) in one phase-slip event to the phase-slip time. In either way, β is a measure of the competition between heat generation and heat evacuation with $\beta > 1$ representing the dominance of heat generation. It is somewhat like the parameter γ used in TDGL equation approach,[13,14] which depends on $\tau_{|\psi|}$ and τ_ϕ.

In order to find the DC *I–V* characteristics of a WL, we need to find the steady state solutions of non-linear Eqs. (3) and (4) and their stability with respect to fluctuations. Two types of steady states are possible: (1) static state where temperature and phase take time-independent values and (2) oscillatory or limit cycle type steady state where $\dot{\phi}$ and temperature oscillate about an average value. We have been guided by numerical solutions, illustrated later for a special case, of the above non-linear equations which eventually led us to analytical solutions for large α values, which is true for most studied devices.

STEADY-STATE SOLUTIONS

For static solutions, $\dot{p} = 0$, and if $p < 1$, $\dot{\phi} = 0$. We have two such solutions. In first case, we get $i_b = i_c(p)\sin\phi$ and $p = p_s = 0$, i.e., $I = I_c(T_b)\sin\phi$ and $T = T_b$. This solution is possible only for $I < I_c(T_b)$ (or $i_b < i_c(0)$). The second static solution is possible only for $T > T_c$, i.e., $p > 1$, and thus $i_c(p) = 0$ and so no super-current flows through the WL. In this case, ϕ is irrelevant and $d\phi/dt \,(= 2\pi V/\Phi_0)$ can be thought in terms of the voltage across the junction. By eliminating $\dot{\phi}$ from Eqs. (3) and (4), we get for static temperature $p_s = \beta i_b^2$, i.e., $T_s = T_b + (I^2 R_N/k)$. The second solution exists only for currents for which $T > T_c$ (or $p > 1$), i.e., $I > I_{sr}$ (or $i_b > i_{sr}$) with

$$I_{sr} = \sqrt{\frac{k(T_c - T_b)}{R_N}} \quad \text{or} \quad i_{sr} = \frac{1}{\sqrt{\beta}}. \qquad (5)$$

We note that the static re-trapping current, I_{sr}, is bath temperature dependent but it is independent of critical current.

From here, we see that when $\beta > 1$, for $I_{sr}(T_b) < I < I_c(T_b)$ at a given T_b, we have two possible static solutions. There exists a cross-over temperature, $T_b = T_h$, at which I_c and I_{sr} are equal (or $i_{sr} = 1$), given by $\beta = 1$. For $T_b < T_h$, $I_c > I_{sr}$, we would get hysteretic IVCs, while for $T_b > T_h$, $I_c < I_{sr}$, we get non-hysteretic IVCs. This T_h is determined by the temperature dependence of I_c. This T_h also brings out the significance of the parameter β with $\beta = 1$ (at T_h) as the cross-over point between the relative magnitudes of heat generation and heat evacuation.

Next, we look for oscillatory steady state (or stable limit-cycle solutions) in the limit of $\alpha \gg 1$. In this limit, we find that the heat generated by a phase-slip event, i.e., $I\Phi_0$, over time τ_J gets evacuated over a much larger time scale τ_{th}. Thus, we make an approximation, which we shall justify later, that the change in temperature, and thus change in i_c, during a phase-slip is negligible. In this limit, we can take p to be fixed while integrating Eq. (3) over one phase-slip happening over time τ_{ps} (in units of τ_J). Using Eq. (3), τ_{ps} is given by

$$\tau_{ps} = \int_0^{2\pi} \frac{d\phi}{2\pi[i_b - i_c(p)\sin\phi]} = \frac{1}{\sqrt{i_b^2 - i_c^2(p)}}. \qquad (6)$$

Using Eq. (3), we also get $\langle\dot{\phi}^2\rangle_{\tau_{ps}} = \frac{1}{\tau_{ps}}\int_0^{\tau_{ps}}\dot{\phi}^2 d\tau = \frac{1}{\tau_{ps}}\int_0^{2\pi}\dot{\phi}d\phi = \frac{2\pi}{\tau_{ps}}\int_0^{2\pi}[i_b - i_c(p)\sin\phi]d\phi = \frac{4\pi^2 i_b}{\tau_{ps}}$. We use this in Eq. (4) to get $\alpha\langle\dot{p}\rangle_{\tau_{ps}} = \alpha\Delta p/\tau_{ps} = -p + \beta i_b\sqrt{i_b^2 - i_c^2(p)}$. Combining with the previously discussed static solutions, we write from Eq. (4)

$$\alpha\langle\dot{p}\rangle_{\tau_{ps}} = -\frac{dU}{dp}$$

$$= \begin{cases} -p & \text{for } i_b \leq i_c(p) \\ -p + \beta i_b\sqrt{i_b^2 - i_c^2(p)} & \text{for } i_b > i_c(p) \quad p \leq 1 \\ -p + \beta i_b^2 & \text{for } p > 1 \end{cases}. \qquad (7)$$

Here, $U(p)$ defines a fictitious potential describing the dynamics of p for whole range of p-values. For steady-state behavior of p, we have to analyze the extremum points of $U(p)$. The minima of $U(p)$ describe the stable steady-states while a maximum separating the two minima gives the sensitivity of the stable steady-state to fluctuations. One can readily see from Eq. (7) that these extremum points are given by $p = 0$, $\beta i_b\sqrt{i_b^2 - i_c^2(p_0)}$, and βi_b^2.

To find the magnitude of temperature oscillations, we recall that the heat evacuated during a single phase-slip event is very small as compared to the heat generated. If we neglect the heat evacuated during one phase-slip event, we can estimate the temperature rise as $\Delta T = I\Phi_0/C_{WL} = i_b\frac{E_J}{C_{WL}} = i_b\frac{\beta}{\alpha}(T_c - T_b)$ or $\Delta p = i_b\beta/\alpha$. This is actually an upper limit as some heat evacuation will happen during phase-slip. So for small values of β/α, the oscillation in temperature will be small. We note that this argument is valid if change in critical current is small due to this small change in temperature. In case the critical current declines very fast with temperature, this approximation will become invalid.

After knowing the steady-state temperature p_0, one can find the IVCs. For $p_0 < 1$, the DC voltage is given by

$$V = \langle V(t) \rangle_{\tau_{ps}} = \frac{\Phi_0}{2\pi\tau_J}\langle \dot{\phi} \rangle_{\tau_{ps}} = \frac{\Phi_0}{2\pi\tau_J}\frac{2\pi}{\tau_{ps}} = R_N I_c(T_b)\sqrt{i_b^2 - i_c^2(p_0)}.$$

For $p_0 \geq 1$, $V = R_N I$. Combining, we get

$$V = \begin{cases} R_N\sqrt{I^2 - I_c^2(T_0)} & \text{for } T_0 < T_c \\ R_N I & \text{for } T_0 \geq T_c \end{cases}. \tag{8}$$

This expression resembles that of the overdamped RCSJ except for the value of T_0, which is dependent on I and T_b.

RESULTS FOR LINEAR $I_c(T)$

Here, we illustrate the above phase and temperature dynamics for short superconducting WLs, where $I_c(T)$ varies linearly with temperature.[1] Thus, we assume $I_c(T) = I_{c0}[1 - (T/T_c)]\Theta[1 - (T/T_c)]$ with Θ as the step-function. Thus, we get $i_c(p) = (1 - p)\Theta(1 - p)$ and $\beta = \beta_0\left(1 - \frac{T_b}{T_c}\right)$ with $\beta_0 = \frac{I_{c0}^2 R_N}{kT_c}$ and $T_h = T_c\left(1 - \frac{1}{\beta_0}\right)$. With this, we can solve

Eqs. (3) and (4) numerically with given initial conditions. Fig. 2 shows the oscillatory steady state time evolution of p and $\dot{\phi}$ for $\alpha = 100$, $\beta = 1.5$ and $i_b = 0.7$. This regime has similarities to the one found by TDGL equations,[13,14] but in our approach, the phase-slip oscillations are driven by the thermal dynamics. This steady state is seen only for certain initials conditions, as discussed later, and for others, we get steady state as $p = 0$ and $\dot{\phi} = 0$. p-oscillates from its mean value of about 0.6115 by an amplitude less than 0.002 justifying our earlier claim of small temperature oscillations for large α values. One can also see that the oscillations are non-sinusoidal and $\dot{\phi}$ and p oscillate in quadrature. This is expected as the rate of change of temperature is maximum at the peak of $\dot{\phi}$, where the heat (power) generated is highest while at the minimum of $\dot{\phi}$ the heat evacuation dominates (see Eq. (4)).

In order to get more general idea about the dynamics of p (and ϕ), we use $i_c(p)$ expression in Eq. (7) to get

$$U(p) = \begin{cases} \dfrac{p^2}{2} + \dfrac{\pi\beta i_b^3}{4} & \text{for } p \leq (1 - i_b) \\[2ex] \dfrac{p^2}{2} + \dfrac{\beta i_b}{2}\left[i_b^2 \sin^{-1}\left(\dfrac{1-p}{i_b}\right) + (1-p)\sqrt{i_b^2 - (1-p)^2}\right] & \text{for } (1 - i_b) < p \leq 1 \\[2ex] \dfrac{p^2}{2} - \beta i_b^2(p - 1) & \text{for } p > 1 \end{cases}. \tag{9}$$

This $U(p)$ depends on T_b through β. We plot $U(p)$ in Fig. 3(a) for $\beta = 1.5$ for selected values of i_b. $U(p)$ has at least one minimum at $p = 0$ for $i_b < 1$, which becomes a maximum if $i_b > 1$. A second minimum exists for $i_b > i_{dr}$ with the dynamic re-trapping current, i_{dr}, given by

$$i_{dr}^2 = (\sqrt{1 + 4\beta^2} - 1)/2\beta^2. \tag{10}$$

For $1 > i_b > i_{dr}$, $U'(p)$ admits two zeroes at $p_\pm = (i^2\beta^2 \pm \sqrt{-i^2\beta^2 + i^4\beta^2 + i^6\beta^4})/(1 + i^2\beta^2)$ other than $p = 0$. Here, p_+ corresponds to the minimum describing a stable limit-cycle while p_- is the maximum of $U(p)$. We plot, in Fig. 3(b), the temperatures, p_s, p_+, and p_-, at which the minima and maxima occur as a function of i_b for various β values. We clearly see a bistable region for $1 > i_b > i_{dr}$.

In order to find the IVCs for a given β_0 value, we have three different segments that describe the steady-state solutions. These are (1) $i_{dr} > i_b > 0$: static superconducting, (2) $i_{sr} > i_b > i_{dr}$: dynamic steady state, and (3) $i_b > i_{sr}$: static resistive. Fig. 4(a) shows the expected IVCs for a constriction with $\beta_0 = 1.5$. We see a non-linear IVC just above I_{dr} as a fraction of current is super-current and above I_{sr} the IVCs become linear. Non-linearity in IVCs is also found in static thermal models due to spread of resistive hot-spot with increasing temperature, however, in that case, no super-current flows above the re-trapping current. We have not captured this static non-linearity due to the simplification

that only the constriction heats up. Fig. 4(b) shows the temperature variation of the three currents, namely I_c, I_{dr}, and I_{sr}, for $\beta_0 = 1.5$. We see that I_c and I_{sr} cross each other at $T/T_c = 1 - 1/\beta_0$. I_{dr} always stays below I_c, and they both go to zero at T_c, but the difference between the two becomes increasingly small as they approach zero.

DISCUSSION AND CONCLUSIONS

$U(p)$ plot in Fig. 3(a) also captures the sensitivity of the two stable points to fluctuations or initial conditions. Thus, when the system is in one of the two minima and if a fluctuation causes it to overcome the maximum separating the two minima, it can transit to the other minimum. The minimum energy required to overcome the maximum so as to go from $p = 0$ minimum to the other is $C_{WL}(T_c - T_b)p_-$, while for reverse process, the system has to loose at least $C_{WL}(T_c - T_b)(p_+ - p_-)$ energy. The barrier may be overcome due to stochastic phase-slip processes.[19] More work is needed to fully capture the role of stochastic fluctuation on superconducting constrictions. Similar to RCSJ model, the fluctuations will give rise to a distribution in the observed critical and re-trapping currents, and in case of large fluctuations, the bistable region may be completely wiped out. In fact, as one approaches T_c, this barrier—between two stable states—becomes smaller while I_c and I_{dr} approach each other, and thus, the hysteresis will disappear—due to fluctuations—before T_c. The maximum (or p_-) also describes the sensitivity
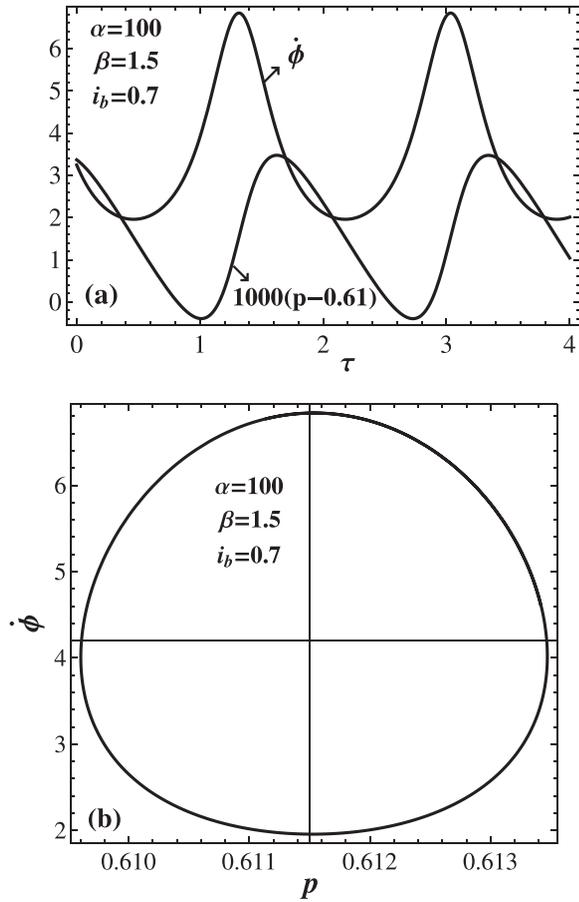
FIG. 2. (a) Oscillatory steady state time evolution of p, actually $1000(p-0.61)$, and $\dot{\phi}$ for $\alpha=100$, $\beta=1.5$, and $i_b=0.7$ found by numerically solving Eq. (3) and (4). (b) shows the steady-state trajectory (limit-cycle) in $p-\dot{\phi}$ space.

FIG. 3. (a) The fictitious potential $U(p)$ for $\beta=1.5$ for various values of i_b. Here, we see that for $i_b < i_{dr} = 0.693$ [see Eq. (10)], there is only one minimum at $p=0$ and for $1 > i_b > i_{dr}$, there are two minima. For $i_b > i_{sr} = 0.816$ [see Eq. (5)], the higher p minimum occurs at $p>1$, i.e., the WL is fully normal. (b) shows the evolution of these minima, p_s (continuous lines), p_+ (dashed lines), and maximum, i.e., p_- (dotted lines), with i_b for various β values.

to the initial conditions for solving Eq. (3) and (4). Thus, if one starts with an initial p-value on the left of the maximum, the system will stabilize in the minimum on the left after a transient. This has been confirmed by numerical solutions.

We also see in Fig. 3(b) that the dynamic region (i.e., $i_{dr} > i_b > i_{sr}$) shrinks as β becomes larger. This will be the case for WLs with high I_c values. For WLs with very small I_c, more precisely, $I_{c0} < kT_c/R_N$ so that $\beta_0 < 1$, the static re-trapping current (I_{sr}) will be smaller than I_c at all temperatures; however, the hysteresis will persist as $I_{dr} < I_c$ [see Fig. 4(b)] at all temperatures. For very large β (or I_c), the dynamic region may be completely suppressed and no supercurrent will exist in the finite voltage branch. Thus, for SQUIDs with constrictions replacing Josephson Junctions, one will see oscillations in the re-trapping current for small β_0 (or I_c) as is the case in some of the experiments[7,8] and not others,[9] where I_c, and therefore β, is large.

The thermal time constant, i.e., τ_{th}, can also be interpreted as a time-scale over which the constriction cools below T_c, and so the superconducting order-parameter is restored over this τ_{th}. This healing time for superconductivity will compete with the other superconductivity recovery time-scales such as $\tau_{|\psi|}$ discussed earlier. This will be relevant in cases where the thermal healing time is small. If τ_{th} is smaller than other time-scales, like $\tau_{|\psi|}$, relevant for recovery of superconductivity in the WL, the hysteresis and detailed I-V characteristics will not be completely dictated by the
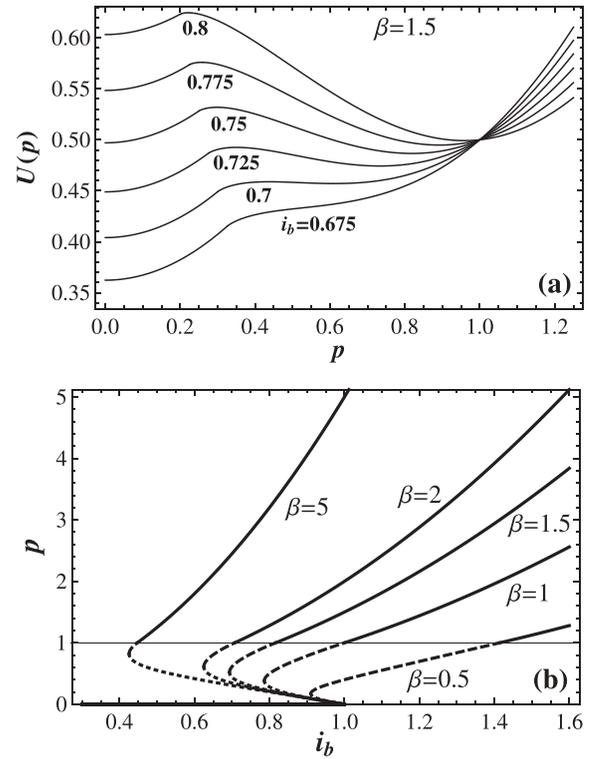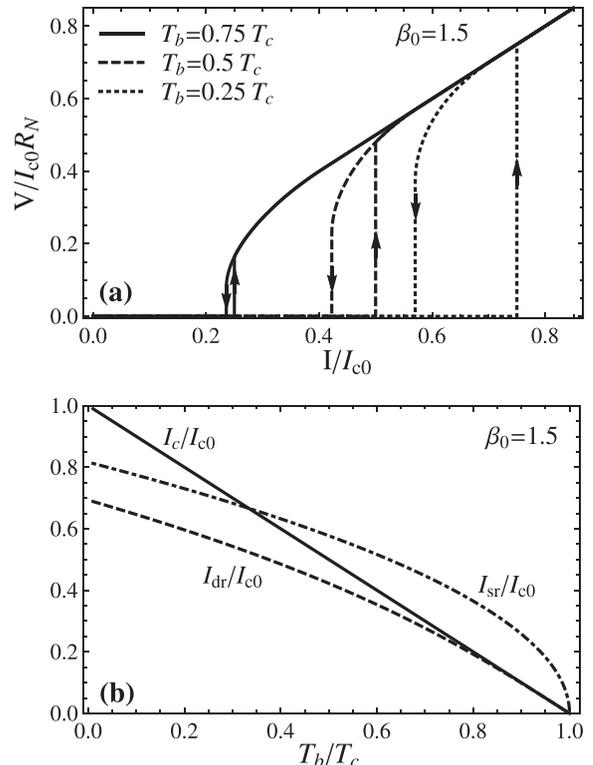

FIG. 4. (a) IVCs for $\beta_0=1.5$ at three different t_b values. (b) Expected variation of I_{dr}, I_{sr} for linear $I_c(T)$ (also shown).

temperature dynamics, and TDGL approach[13,14] may be more appropriate in such cases. If τ_{th} dominates the other time scales, the superconducting order parameter cannot recover faster than τ_{th}, and thus, TDGL alone cannot describe the order-parameter dynamics and I-V characteristics. In such a case, the thermal dynamics alone may be sufficient to deduce the I-V characteristics of the WL.

In conclusion, we have illustrated the deterministic dynamics of temperature and phase in a superconducting constriction using a simple time dependent thermal-model. A new dynamic region is found where the bias-current is shared dynamically between super-current and normal current giving rise to a finite voltage. The Joule heat thus generated in the constriction region raises its temperature but not above T_c in this regime. A new bi-stable region is found between I_{dr} and I_{sr}. The oscillations of re-trapping current in some of the WL based SQUIDs seen in earlier reported experiments[7,8] is consistent with this dynamic regime.

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